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# HILBERT ALGEBRAS

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**1.** Since the publication of the papers 'On Rings of Operators' by F. J. Murray and J. von Neumann, there has been an increasing interest among mathematicians in the structure of topological algebras. Today I wish to discuss some properties of a class of algebras called Hilbert algebras.

Hilbert algebras are defined by means of the following axioms:

A: **A** is a * algebra over the field of complex numbers in addition to being an inner product space: that is to say

$A_1$: **A** is a linear vector space over the field $C$ of complex numbers.

$A_2$: **A** is a ring where multiplication is associative and distributive with addition, but not necessarily commutative.

$A_3$: There exists an involution operation denoted by *, which associates to each element $a \in \mathbf{A}$, a unique element $a^* \in \mathbf{A}$ such that:

($\alpha$) $(a^*)^* = a$,
($\beta$) $(\lambda a)^* = \bar{\lambda} a^*$, $\lambda \in C$, $\bar{\lambda}$ complex conjugate of $\lambda$,
($\gamma$) $(a+b)^* = a^* + b^*$,
($\delta$) $(ab)^* = b^* a^*$,
($\epsilon$) $xx^* \neq 0$ unless $x = 0$.

$A_4$: **A** is an inner product space: that is, to each pair of elements $a$, $b$ in **A** we can associate a scalar denoted by $(a, b)$ which is linear in $a$ and conjugate linear in $b$, satisfying the following conditions:

($\alpha$) $(\lambda a + \mu b, c) = \lambda(a, c) + \mu(b, c)$, $\lambda, \mu \in C$,
($\beta$) $(a, b) = \overline{(b, a)}$,
($\gamma$) $(a, a) \geqslant 0$ for all $a$ in **A** and is zero if and only if $a = 0$,
($\delta$) $(ab, c) = (a, cb^*) = (b, a^*c)$,
($\epsilon$) $(a, b) = (b^*, a^*)$.

We shall consistently use the following notations:

$\|x\| = \sqrt{(x, x)}$, the positive value of the square root being taken.

Greek letters will denote complex numbers and Roman letters (small type) will denote elements of the algebra **A** or its completion.

Now **A** may or may not be complete under the Hilbert norm topology. If **A** is already complete it is natural to assume that multiplication is continuous with respect to both the variables; in which case we have

(*) (A continuity), Algebra is complete and $|(a, b)| \leq \|a\| \|b\|$.

Ambrose's **H***-algebra whose structure is well known [cf. Loomis, *An Introduction to Abstract Harmonic Analysis*].

If **A** is not complete, we denote by $\mathfrak{h}$ the completed Hilbert space. Then, following classical procedure, the mappings $x \to ax$ and $x \to xa$ can be defined for all $x$ in $\mathfrak{h}$ and $a$ in **A** so that they are continuous for fixed $a \in \mathbf{A}$, throughout $\mathfrak{h}$. But we cannot say anything about the existence of $xy$ or $yx$ when both $x$ and $y$ are in $\mathfrak{h}$ but outside **A**. So we add the following axiom B to axiom A.

B. If **A** is not complete, the operators $x \to ax$ and $x \to xa$ are so extended to $\mathfrak{h}$ that they are closed linear operators in $\mathfrak{h}$, and these operators are bounded if and only if $a \in \mathbf{A}$.

Axiom B implies that the product $xy$ certainly exists if at least one of the two elements $x$, $y$ belongs to **A**. As an immediate consequence of these axioms one readily proves that the * operation is a bounded operation which can be extended all over $\mathfrak{h}$, so that to each element $x \in \mathfrak{h}$ there corresponds $x^*$ so that $\|x\| = \|x^*\|$ and axiom $A_3$ is fulfilled. Further the linear operators $x \to ax$, $x \to xa$, $x \to a^*x$, $x \to xa^*$, for $a \in \mathbf{A}$ have the same bounds and their common bound, called the *uniform norm* of $a$, is denoted by $|||a|||$.

$$|||a||| = \underset{x \in \mathfrak{h}}{\text{l.u.b.}} \frac{\|ax\|}{\|x\|}.$$

With respect to the uniform norm, **A** will be in general an incomplete Banach space, unless **A** contains the unit of multiplication.

There is still another axiom which is of interest, though not essential for our discussion, namely:

C. Whenever $a$ and $b$ are two elements in $\mathfrak{h}$, if there exists a third element $c$ and a Cauchy sequence $b_n$, such that

$$b_n \to b$$

and $$(ab_n, x) \to (c, x) \quad \text{for every } x,$$

then and only then $ab$ exists and $c = ab$.

This axiom assures us that the adjoint of the linear operator $x \to ax$ is $x \to xa^*$ for any $a \in \mathfrak{h}$. It is obvious that axiom C is of significance only when $a \in \mathfrak{h}$ and is outside **A**.

**2.** The set **A** together with its completion $\mathfrak{h}$ satisfying axioms A and B is called a Hilbert algebra. Our purpose is to discuss the structure of this algebra. We introduce a few definitions and notations which we need.

(*a*) An element $x \in \mathfrak{h}$ is called *self-adjoint* if $x = x^*$.

(*b*) An element $x$ is said to be *positive* if $x = x^*$ and $(xy, y) \geqslant 0$ for every $y \in \mathbf{A}$.

(*c*) An element $e$ is called an *idempotent* if $e^2$ exists and $e^2 = e$. If $e$ is also self-adjoint we call $e$ a *self-adjoint idempotent.*

One easily verifies that a self-adjoint idempotent is invariably an element in $\mathbf{A}$.

(*d*) If $S$ is a subset of $\mathfrak{h}$, we denote by $[S]$ its linear closure.

(*e*) A closed linear manifold $\mathscr{M} \subset \mathfrak{h}$ is called a *right ideal manifold*, if whenever $x \in \mathscr{M}$, $[x\mathbf{A}] \subset \mathscr{M}$. A similar definition holds for *left ideal manifolds* and (two sided) *ideal manifolds*. The right ideal manifold $[a\mathbf{A}]$ is called a *principal right ideal manifold* and $a$ the *generating element* of it.

One easily verifies that $[e\mathbf{A}] = [e\mathfrak{h}] = e\mathfrak{h}$.

(*f*) The Hilbert algebra $\{\mathbf{A}, \mathfrak{h}\}$ is called *simple* if $\mathfrak{h}$ does not contain any (two-sided) ideal manifolds other than the trivial ones. The algebra is said to be *abelian* if multiplication is commutative.

(*g*) There are two extensions of the algebra $\mathbf{A}$ to the algebra of linear operators in $\mathfrak{h}$ viz. those linear operators $L$ which satisfy the condition

$$L(xy) = L(x)\,y,$$

the mapping $x \to ax$ being denoted by $L_a(x)$, and those operators $R$ which satisfy the condition

$$R(xy) = xR(y),$$

the mapping $x \to xa$ being denoted by $R_a(x)$. The totality of operators of the type $L$ will be denoted by $\mathscr{L}(\mathbf{A})$ and those of the type $R$ will be denoted by $\mathscr{R}(\mathbf{A})$.

One verifies that the commutant of $\mathscr{L}$ is $\mathscr{R}$ and of $\mathscr{R}$ is $\mathscr{L}$.

That in the algebra $\mathbf{A}$ there are sufficiently many self-adjoint idempotents can be proved, in the usual way, by taking positive elements $a$ in $\mathbf{A}$, polynomials $p(a)$, and their limits, in the Hilbert norm.

**3.** If the algebra $\mathbf{A}$ is abelian then $\mathfrak{h}$ will be isometrically isomorphic to square integrable functions on a locally compact Hausdorff space $S$ with a measure $\nu$, multiplication being defined as the ordinary multiplication of two functions in $S$. $\mathbf{A}$ will be isomorphic to bounded functions vanishing at $\infty$. This can be proved either using Gelfand's isomorphism theorem for commutative Banach algebras [cf. Dixmier, *Les algèbres d'opérateurs dans l'espace hilbertien*, 1957] or by observing that the self-adjoint idempotents in $\mathbf{A}$ form a measure algebra and using Stone's theorem [Halmos, *Measure Theory*]. If $\mathbf{A}$ contains the unit of multiplication, $S$ will be compact.

4. If the algebra $\mathbf{A}$ is simple, that is to say there are no (two sided) ideal manifolds in $\mathfrak{h}$ other than $\{0\}$ and $\mathfrak{h}$, then (and only then) $\mathscr{L}(\mathbf{A})$ and $\mathscr{R}(\mathbf{A})$ will be factors. The self-adjoint idempotents in $\mathbf{A}$ will satisfy the axioms of continuous geometry of von Neumann, in the sense that there exists an equivalence relation among them and the classes of equivalent idempotents form a well-ordered set with a dimension number attached to the self-adjoint idempotent $e$, viz. $d(e) = \|e\|^2$. There are only four possible cases, viz. the four types of factors of von Neumann $I_n$ ($n$ a finite positive integer), $II_1$, $I_\infty$, $II_\infty$. If $\mathbf{A}$ contains a unit and is simple, the types $I_n$, $II_1$ will occur, and the types $I_\infty$ and $II_\infty$ will occur when $\mathbf{A}$ does not contain a unit. $I_n$ and $I_\infty$ are called discrete while $II_1$ and $II_\infty$ are continuous types. $I_n$ and $I_\infty$ will occur if the simple algebra $\mathbf{A}$ contains minimal idempotents, and $II_1$ and $II_\infty$ otherwise.

It is verified easily that $I_n$ is isomorphic to the algebra of matrices of order $n$ with complex numbers as their elements, and $I_\infty$ to the totality of bounded linear operators in a Hilbert space. The structure of $II_\infty$ can be determined if we know the structure of $II_1$.

The structure of a simple algebra of type $II_1$ seems to be difficult and intriguing. Von Neumann has constructed two types of examples of an algebra of type $II_1$. If $\mathbf{A}$ is a simple algebra of type $II_1$ then $\mathbf{A}$ contains a unit, whose norm we shall assume to be 1, and there are no minimal self-adjoint idempotents. It is likely that the structure of $\mathbf{A}$ can be determined in terms of its maximal abelian subalgebras. It is quite likely the maximal abelian subalgebras of a simple algebra are unitarily equivalent. In the case of $I_n$ the diagonal matrices form a maximal abelian subalgebra and it is well known that all maximal abelian subalgebras are unitarily equivalent to the subalgebra of diagonal matrices. In the case of an algebra $\mathbf{A}$ of type $II_1$, let $\mathbf{B}$ be a maximal abelian subalgebra and $\overline{\mathbf{B}}$ its Hilbert completion. If we observe that the dimension numbers of the self-adjoint idempotents in $\mathbf{B}$ take all values between 0 and 1, then one can easily show that all maximal subalgebras are unitarily equivalent.

5. If the algebra $\mathbf{A}$ is neither simple nor abelian then one has to study the structure of $\mathbf{A}$ in relation to its centre. Let $\mathbf{C}$ denote the centre of $\mathbf{A}$ and let $\mathbf{Z}$ be its Hilbert completion. A self-adjoint idempotent $e$ in the centre is said to be minimal if it cannot be expressed as sum of two idempotents in the centre. If $e$ is minimal $e\mathbf{A}$ is a simple algebra with $e\mathfrak{h}$ as its Hilbert completion. $\mathbf{C}$ being abelian, there exists a locally compact (or compact) Hausdorff space $S$ and a measure $\nu$ so that $\mathbf{Z}$ is isomorphic

to $L_2$ functions on $S$. The atomic elements in $S$ will correspond to minimal idempotents in $\mathbf{C}$, so that $\mathbf{A}$ can be expressed as a direct sum of two algebras $\mathbf{A}_1$ and $\mathbf{A}_2$, that is

$$\mathbf{A} = \mathbf{A}_1 + \mathbf{A}_2,$$

$$\mathfrak{h} = \mathfrak{h}_1 + \mathfrak{h}_2, \quad [\mathbf{A}_1] = \mathfrak{h}_1,\ [\mathbf{A}_2] = \mathfrak{h}_2,$$

$\mathfrak{h}_1$ and $\mathfrak{h}_2$ being orthogonal manifolds, where $\mathbf{A}_1$ is a direct sum of simple algebras and $\mathfrak{h}_2$ contains no minimal ideal manifolds. Further decomposition of $\mathfrak{h}_2$ is possible along the lines of von Neumann's reduction theory, but we shall not discuss it here.

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